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**Evidence for a hyperbolic-like distribution of asset returns drawn
from a simple economical financial markets model**

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Abstract

Risk management and asset pricing benefit from simple functional descriptions of the distribution of real asset returns. Recently, several authors have proposed that asset returns in real stock markets are distributed according to a hyperbolic distribution. While asset returns are generated by trades over time, the natural question is: What does economic theory imply concerning return distributions? We propose a simple model of price formation and, thus, return distribution which is based on economic reasoning. The markets behavior is represented by a pair consisting of a time-constant strategy and a dynamical trading strategy generating a flow between funds. Simulations of the price dynamics generate returns with fat-tail behavior in line with that of a hyperbolic distribution, and also volatility clustering, which is a mayor stylized fact of asset returns.

Keywords: Asset returns, hyperbolic distribution, evolutionary finance

JEL numbers: G12, C51

1 Introduction

Prices are observable entities of a financial market. With a view towards risk management and asset pricing, numerous distributions to model the returns have been proposed. For a broad overview over this topic see the monograph by Bouchard and Potters (2000). Among others, such as ease of computation and estimation, the value of a proposed distribution is related to the degree to which it fits to the data. Having found a good distribution, the next question is: Which stochastic process generates this distribution? This stochastic process is then taken as the model for the process under consideration. This model is purely descriptive rather than explanatory. Barndorff-Nielsen (1977) introduced the family of (generalized) hyperbolic distributions. While his original concern was the particle size distribution in certain transport phenomena, several authors including Barndorff-Nielsen, Eberlein and Keller, and Bibby and Sørensen used this family of distributions successfully for modelling stock prices. For an overview about diffusion models generating (generalized) hyperbolic densities, see Bibby and Sørensen (1997) and Sørensen (2003) and the references therein. Prices are generated during trades between agents on the financial market. Therefore the distribution of asset returns should be deducible from economic reasoning, if one would know the mechanism by which prices are generated over time. The problem here is that there is not a unique, well-established, and agreed model for price formation. Apart from its details, (mainstream) theoretical economics seems to at least agree that the price process is a multiplicative random walk, in which the multiplicative factor is a random variable, see Duffie (2001). Details of the distribution of this random variable are left to specific modelling and speculation. The aim of this note is not to propose some functional form of a price distribution that fits real data, but rather to deduce a distribution of (log-) returns from an elementary model that takes into traders' actions on the market.

2 On the hyperbolic distribution

To fix notation: Given the price trail $Q(t)$ of an asset its (log) return is defined as

$$S(t, \Delta t) := \log Q(t + \Delta t) - \log Q(t), \quad \Delta t > 0$$

while the absolute return is $|S(t, \Delta t)|$. The distribution of returns is defined by

$$P_{\Delta t}(Z) = \Pr[S(t, \Delta t) = Z].$$

(For finite trails lengths, P is actually the relative frequency.) With respect to tail behavior it is worthwhile to consider the semi-logarithmic plot of the distribution $(Z, \ln P(Z))$. The distribution $P_{\Delta t}$ of returns $S(t, \Delta t)$ is non-Gaussian, in that it is more sharply peaked than the normal

distribution, while its tails are more heavy; see Figure 1. There is no consensus about which distribution fits financial return data best. Various parametric models have been discussed in the literature, see Cont (2001) and the references therein.

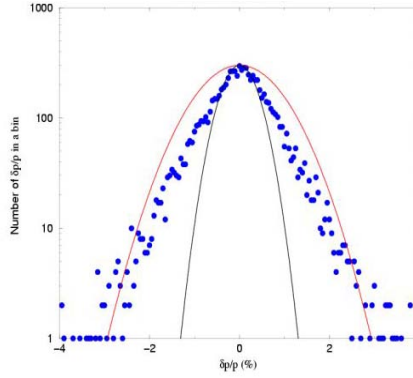


Figure 1: *Histogram of return distribution for daily returns of the DJIA from 1966 - 1989. Solid lines represent different Normal distributions.*

In a semi-logarithmic plot, the return distribution looks triangular rather than parabolic, see Figure 1. This finding supports the conjecture by Madan and Senata (1990) that the observed return distribution belongs to the family of hyperbolic distributions. Stronger evidence from data is discussed later. Roughly speaking, a hyperbolic distribution is characterized by the fact that the logarithm of its density forms a hyperbola. The density of the hyperbolic distribution $P(x) = P(x; p, q, \delta, \mu)$ is such that

$$\ln P(x) \sim \text{const} - p\sqrt{\delta^2 + (x - \mu)^2} + q(x - \mu)$$

with four parameters denoted by (p, q, δ, μ) , where $\mu \in \mathbb{R}$ and $0 < \delta \in \mathbb{R}$ serve as location and scale, respectively, and p, q are the shape parameters of the distribution. The range of the shape parameters is a triangle

$$\Delta = \{(p, q) \in \mathbb{R}^2 : 0 \leq |q| < p < 1\}.$$

Hence, any hyperbolic distribution can be uniquely represented by a point $(p, q) \in \Delta$; see Figure 2. This family includes the normal and the (possibly) skew Laplace distribution, particularly in the limit $p \rightarrow 0$, the distribution approaches the normal density, while for $(1, 0)$ it coincides with the (location-scale) Laplace. Further details about generalized hyperbolic distribution can be found in Eberlein and Prause (2002) and Sørensen (2003) and will be omitted here.

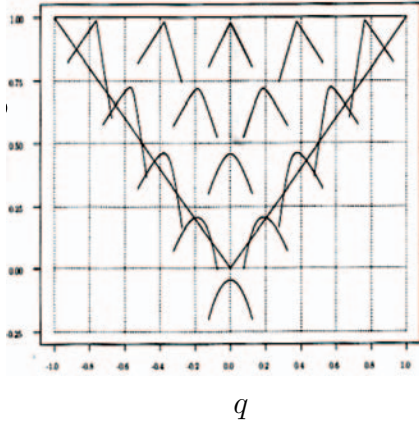


Figure 2: *Hyperbolic log densities with mean 0 and variance 1 for different parameter values of p and q , from Bibby and Sørensen (1997)*

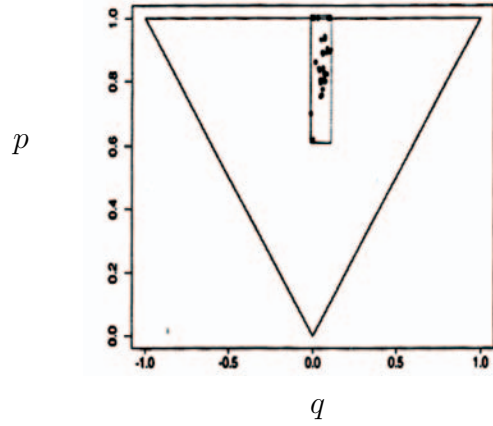


Figure 3: *Shape triangle: Estimated parameters for several stocks, see Kücher et al. (1999). The region of estimated parameters is narrow.*

Madan and Seneta (1990) proposed that the distribution of asset returns is hyperbolic. Eberlein and Prause (2002) and also Kücher et al. (1999) estimated the two shape parameters of the hyperbolic distributions for a number of asset returns and found that these parameters lie in a quite narrow region close to $q = 0$, see Figure 3. In fact the estimated values of q are only slightly larger than 0 and, hence the distribution is slightly skew due to a small trend in weekly data and, more interestingly, the estimated range for p is $[0.6, 1]$. This corresponds to the fact that returns are non-Gaussian ($p \gg 0$), but are roughly triangular, i.e., their shape – in a logarithmic plot – ranges from a tent with a sharp central edge ($p = 1$) to a "soft" tent with a smooth central edge ($p < 1$). Hence, the data seem to support the conjecture that asset returns are hyperbolic.

The conjecture by Madan and Seneta was purely data driven and has no economic explanatory component. Our question is whether this conjecture can be justified from economic reasoning. We propose a model from traditional finance (see Blume and Easley (1992) and also Evstigneev, Hens, and Schenk-Hoppé (2002)), in which prices are generated by trades between agents on the market. The set of market participants is partitioned into classes in which all investors share the same strategy. An agent is a representative of the strategy class to which it belongs. Different agents compete for money income, measured as the relative market share of the agent. Generally, a strategy is a mixture of basic investment rules. This mixture can vary in time, i.e., during time, an agent may change his partitioning of relative wealth among

different investment styles (funds). This decision is thought to depend on some economic trigger. This creates a flow of funds for each agent. We consider the following situation: An agent faces a market – thereby the agent follows a dynamical strategy while the market’s behavior is represented by a time-constant strategy. This accounts for the fact that an individual agent may change its strategy more rapidly than the market’s aggregate of strategy mixtures changes.

3 The fundamental economic model of wealth dynamics

In this section, we propose an elementary model for endogenous price formation by trade, based on Evstigneev, Hens, and Schenk-Hopp (2002). The most fundamental idea is the following: A financial market is established by a huge number of traders, such as individuals and institutional investors. This collection is partitioned into a set of trading strategies, each strategy representing the set of all investors following that strategy. A trading strategy consists of two settings: one is the investment style and the other one is called the allocation style. The investment style determines the amount of wealth that is spent on the financial market, while the allocation style determines how much of this invested wealth is spent across assets. In the following, we assume that all investors have the same investment style. Trades, i.e. re-allocation of resources, can be regarded as the result of an interaction between trading strategies. Thus the strength of these interactions reflects the relative impacts of competing strategies. The more wealth a strategy has collected during the trading process, the higher is its impact. Wealth earned by a trading strategy depends on the dividends paid by the assets selected. Therefore, dividends are one of the driving forces for this process.

3.1 The fundamental allocation styles

The financial market is characterized by the set of $K \geq 2$ available assets \mathcal{K} and a given set of basic allocation styles $\mathcal{L} = \{\boldsymbol{\lambda}^\alpha, \alpha \in \mathcal{A}\}$, where \mathcal{A} is a finite alphabet of cardinality $|\mathcal{A}| > 0$ and $\|\boldsymbol{\lambda}^\alpha\|_1 = 1$. Imposing short-sale constraints, we require that $\boldsymbol{\lambda}^\alpha > 0$, which is to say that $\lambda_k^\alpha \geq 0$ and there is some k such that $\lambda_k^\alpha > 0$, i.e., we assume that each fundamental allocation has positive demand in at least one asset.

Let $\Delta_{\mathcal{L}}$ denote the simplex spanned by \mathcal{L} . Each strategy in $\Delta_{\mathcal{L}}$ on the market can be represented as a convex combination of basic allocation styles $\boldsymbol{\lambda}^\alpha$, while the mixture may vary over time:

$$\boldsymbol{\lambda}_t := \sum_{\alpha} \ell_{\alpha}(t) \boldsymbol{\lambda}^{\alpha}, \quad \ell_{\alpha}(t) \geq 0, \quad \sum_{\alpha} \ell_{\alpha}(t) = 1.$$

Thus each strategy $\lambda_t \in \Delta_{\mathcal{L}}$ is uniquely represented by a vector $\ell(t) = (\ell_{\alpha}(t))_{\alpha}$. Then the trajectory $\ell = (\ell(t))_t$ can be regarded as a flow between basic allocation styles \mathcal{L} . These flow

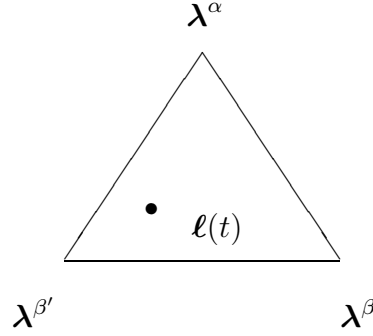


Figure 4: *Simplex of strategies on the market. The strategy mix on the market is represented by a point $\ell(t) \in \Delta_{\mathcal{L}}$*

dynamics are thought to depend on some observable economic variable, denoted by \mathbf{x} . This trigger represents available information, externalities, or other observables of the market itself. By a slight abuse of notation, we may also write

$$\ell_{\alpha}(t) = \ell_{\alpha}(\mathbf{x}_t)$$

to express the fact that the coefficient of λ^{α} is some given function of the trigger variable \mathbf{x}_t , where $\ell_{\alpha} : \mathbb{R}^n \rightarrow [0, 1]$, for some n . Then the portion which is invested in asset k according to allocation style α given \mathbf{x}_t is

$$\lambda_{t,k} = \sum_{\alpha} \ell_{\alpha}(\mathbf{x}_t) \lambda_k^{\alpha}.$$

3.2 The basic value process

Let $\mathcal{E} = (\mathcal{K}, \mathcal{L})$ be a finite financial market as described before. Uncertainty of the market is represented as usual: $(\Omega, \mathcal{S}, \mathcal{F})$, is a given probability space (Ω the set of states, \mathcal{S} a σ -algebra of events, \mathcal{F} some probability measure), which is equipped with some filtration $\mathbb{S} = [\mathcal{S}_0, \dots, \mathcal{S}_T, \dots]$. We assume that the price process \mathbf{Q} as well as the dividend process are stochastic processes adapted to the filtration \mathbb{S} in that for each t , \mathbf{Q}_t and \mathbf{D}_t are random variables with respect to (Ω, \mathcal{S}_t) . There are J different strategies on the market, where $\lambda_t^i = (\lambda_{t,k}^i, k \in \mathcal{K})$ is the strategy $i \in \mathcal{I}$ defined by

$$\lambda_t^i = \sum_{\alpha} \ell_{\alpha}^i(\mathbf{x}_t) \lambda^{\alpha}.$$

If w_t^i is the wealth of the (individual) strategy i at time t , then the portfolio bought on the market is $\theta_t^i = (\theta_{t,k}^i, k \in \mathcal{K})$, where, according to the strategy i ,

$$\theta_{k,t}^i = \frac{\lambda_{t,k}^i w_t^i}{Q_{t,k}},$$

and where $Q_{t,k}$ is the price of one unit of asset k on the market at time t . Note that we assume that assets are infinitely divisible. This assumption is an acceptable approximation on a large market.

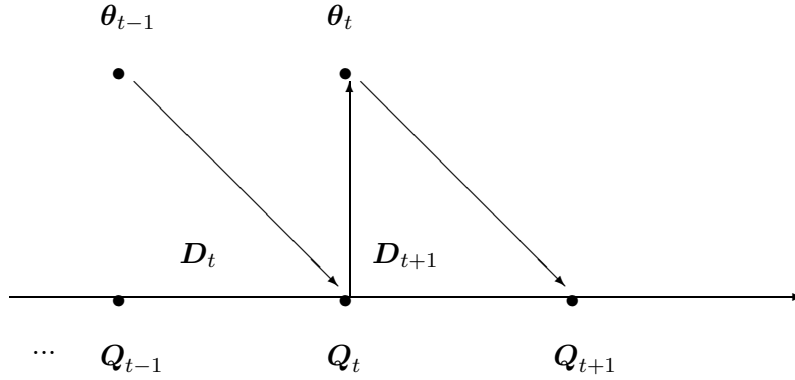


Figure 5: *Portfolio transformation in a time-discrete Multi-Period model, with short hand notation $\mathbf{D}_{t+1} = \mathbf{D}_{[t,t+1]}$ is the dividend paid during the interval $[t, t+1]$.*

Time is partitioned into equally long, half-open intervals $[t, t+1)$, while trade happens only in t while dividends are paid at the end of the period, i.e. \mathbf{D}_t is the vector of dividend paid during the period $[t-1, t)$. The state of a financial market at some time t is assumed to be completely characterized by the vector of asset prices \mathbf{Q}_t and the vector of dividends \mathbf{D}_t .

Assume that strategy i has some initial wealth w_0^i , which is completely invested into a portfolio $\theta_0^i = (\theta_{k,0}^i)$ defined by

$$\theta_{k,0}^i = \frac{\lambda_{0,k}^i}{Q_{0,k}} w_0^i.$$

The investment at time 0 therefore is $\mathbf{Q}_0 \theta_0^i = \sum_k \lambda_{0,k}^i w_0^i = w_0^i$. Its payoff at the end of the interval $[0, 1)$ is \mathbf{D}_1 , while its capital gain on the market at time 1 is $\mathbf{Q}_1 \theta_0^i$. At time 1, a new portfolio is bought for $\mathbf{Q}_1 \theta_1^i$. The wealth w_{t+1}^i at time $t+1$ after reinvestment into the new

portfolio θ_{t+1}^i therefore is

$$\begin{aligned} w_{t+1}^i &= (D_{t+1} + Q_{t+1})\theta_t^i - Q_{t+1}\theta_{t+1}^i \\ &= D_{t+1}\theta_t^i - Q_{t+1}\dot{\theta}_t^i, \end{aligned}$$

where $\dot{\theta}_t^i := \theta_{t+1}^i - \theta_t^i$. The entity $\Theta_t := \sum_i \dot{\theta}_t^i$ can be considered as the vector of net trades, while $\|\Theta_t\|_1$ can be regarded as the volume traded.

We assume that all agents follow the same "investment mode" characterized by

$$Q_{t+1}\dot{\theta}_{t+1}^i = 0 \quad \text{for all } i. \quad (1)$$

This assumption can be called self-financing, see also Appendix A. Under this assumption, the wealth accumulation in the bank account belonging to strategy i yields $w_{t+1}^i = D_{t+1}\theta_t^i$ and thus we obtain the fundamental evolution equation for this investment rule as

$$w_{t+1}^i = w_t^i \sum_k \frac{D_{t+1}^k}{Q_t^k} \lambda_{t,k}^i. \quad (2)$$

To normalize this expression, first define the total wealth collected by all strategies,

$$W_{t+1} = \sum_k D_{t+1}^k S_{t,k},$$

where S_t is the vector of exogenous supplies. One then obtains for the relative wealth $r_t^i := w_t^i/W_t$

$$\begin{aligned} r_{t+1}^i &= r_t^i \sum_k \frac{D_{t+1}^k S_t^k}{\sum_{k'} D_{t+1}^{k'} S_{t,k'}} \frac{W_t}{S_t^k Q_t^k} \lambda_{t,k}^i \\ &= r_t^i \sum_k d_{t+1}^k \frac{W_t}{Q_t^k S_t^k} \lambda_{t,k}^i, \end{aligned}$$

where $d_{t+1} = \left(\frac{D_{t+1}^k S_t^k}{\sum_{k'} D_{t+1}^{k'} S_{t,k'}} \right)_k$ is the vector of relative dividends paid at time $t+1$. Note that $0 \leq r_t^i$, while $\sum_i r_t^i = 1$. Let

$$Q_t^k := \frac{W_t}{S_t^k} p_t^k,$$

where $p_t^k = p^k(\Lambda_t, \mathbf{r}_t)$ is the relative price of the k^{th} asset, which may depend not only on the demand for this asset but also on other factors. Then

$$r_{t+1}^i = r_t^i \left(\sum_k \frac{d_{t+1}^k}{p_t^k} \lambda_{t,k}^i \right), \quad (3)$$

where $y_{t+1}^k := d_{t+1}^k/p_t^k$ can be regarded as the corresponding (relative) dividend yield.¹ Inserting $\lambda_t^i = \sum_{\alpha} \ell_{\alpha}^i(t) \lambda^{\alpha}$, we obtain

$$r_{t+1}^i = r_t^i \sum_{\alpha} \ell_{\alpha}^i(t) \langle \mathbf{y}_{t+1}, \lambda^{\alpha} \rangle,$$

where we define $\mathbf{y} = \left(\mathbf{y}^{(\alpha)} \right)_{\alpha}$ with $\mathbf{y}^{\alpha} := \langle \mathbf{y}_{t+1}, \lambda^{\alpha} \rangle$ being the projection of the vector of relative dividend yields on the allocation style λ^{α} , so that

$$r_{t+1}^i = r_t^i \sum_{\alpha} \mathbf{y}_{t+1}^{(\alpha)} \ell_{\alpha}^i(t) = r_t^i \left(\mathbf{y}_{t+1} \mathbb{L}^i(t) \right),$$

where $\mathbb{L}(t) = \left(\ell^i(t) \right)_{i=1..I}$ is the matrix of strategies. Hence we obtain

$$\mathbf{r}_{t+1} = \mathbf{r}_t \star \left(\mathbf{y}_{t+1} \mathbb{L}(t) \right),$$

where \star denotes componentwise multiplication. Since $\|\mathbf{r}_t\|_1 = 1$ for all t , we have the condition that

$$1 = \sum_{\alpha} r_t^i y_{t+1}^{\alpha} \ell_{\alpha}^i(t) = \sum_{i,k} d_{t+1}^k \frac{r_t^i \lambda_{t,k}^i}{p_{t+1}^k}$$

This condition implicitly determines possible pricing rules $p_t^k(\mathbf{r}, \lambda)$. One particular solution to this equation is

$$p_{t+1}^k = \sum_j r_t^j \lambda_j^k. \quad (4)$$

Due to equation (4) prices are weighted averages of the strategies on the market.

4 An elementary model of a market

To make things as simple as possible, we consider a complete financial market with only two funds. According to our model, the market's behavior is represented by strategies. Each strategy can be regarded as a class of trading rules. The fundamental assumption is that the markets behavior can be described by two (strategic) components: a time-constant or slow varying component and a fast varying component. These two classes of strategies compete on the market for wealth and so generate a price process. In contrast to Lux (1998), for example, we do not consider a market as a multi-agent system, rather than study the behavior of the interplay of two dynamical market components: a slow varying (adiabatic) one and a fast varying. As basic allocation styles we choose the canonical one, i.e.,

$$\lambda^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (5)$$

¹Often the term relative dividend yield is restricted to the situation where supply is normalized to 1.

With respect to the canonical basis (5), the slow (constant) strategic component on the market is represented by a simple strategy with parameter α

$$\boldsymbol{\lambda}^M = \alpha \boldsymbol{\lambda}^1 + (1 - \alpha) \boldsymbol{\lambda}^2,$$

while the dynamical strategic component is given by

$$\boldsymbol{\lambda}_\beta(t) = \ell_\beta(t) \boldsymbol{\lambda}^1 + (1 - \ell_\beta(t)) \boldsymbol{\lambda}^2.$$

where $0 < \ell(t) < 1$ and β is some parameter, see below. In the following we also denote the constant strategy by $[\alpha]$. For $\ell(t)$ we make the following *ansatz*

$$\ell_\beta(t) = \ell_\beta(x_t)$$

The corresponding strategy $\ell_\beta(t)$ is also denoted by (β) in the following. We suppose that the flow is generated by a trigger x_t which is some function of the respective asset demands $y_{t+1}^k = d_t^k / p_{t+1}^k$. A particular simple trigger is the excess demand

$$x_t = y_{t+1}^1 - y_{t+1}^2.$$

A potential choice for the strategy (β) is

$$\ell_\beta(x_t) = \frac{e^{\beta x_t}}{1 + e^{\beta x_t}}, \quad (6)$$

so that $\ell_\beta(t)$ is a flow between $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$. This particular form was chosen for the sake of simplicity and may be economically motivated in terms of Prospect Theory, see Kahneman and Tversky (1979). The graph of the function ℓ_β is a sigmoid in the range $[-\infty, \infty]$ and takes values in $[0, 1]$, while it is strictly increasing if $\beta > 0$ and strictly decreasing if $\beta < 0$. The parameter β serves as a measure for the sensitivity with which the portfolio composition reacts to changes in x in that $\left. \frac{d}{dx} \ell_\beta(x) \right|_{x=0} = \frac{\beta}{4}$. We therefore call β the *elasticity* of the strategy. $x > 0$ if asset 1 has a higher return than asset 2. Therefore, given that $\beta > 0$, $x > 0$ leads to more demand for asset 1. The corresponding strategy (β) , $\beta > 0$, can therefore be described as a strategy according to which wealth flows towards the fund with the better performing asset. If $\beta < 0$, then the strategy (β) is to buy the worse asset. Thus, the case $\beta > 0$ might be regarded as a simple trend follower strategy, while the case $\beta < 0$ might be seen as a mean reverting strategy. Lux and Marchesi (1999) studied a financial market as a large multi-agent system, in which agents can follow different strategies, i.e. they are noise traders or fundamentalists. Noise traders may rely on non-fundamental sources of information such as price trends. In our setting, a strategy (β) with parameter $\beta < 0$ can be regarded as a fundamentalist's strategy, while (β) with $\beta > 0$ can be regarded as a noise trader's strategy relying on the idea of momentum reverting.

5 Numerical estimates

Recall the finding displayed in Figure 3. There, distributions of real asset returns were assumed to belong to the family of hyperbolic distributions with shape parameters (q, ξ) . One sees that the parameter range is quite small, $q \approx 0$, and $\xi \in [0.6, 1]$. Thus the observed distribution is nearly symmetric and ranges from a soft to a hard tent. Our model of a market can reproduce this qualitative finding. Relative dividends are assumed to be uniformly distributed in $(0, 1)$ while $d_t^1 + d_t^2 = 1$ for all t , while the two strategies $[\alpha]$ and (β) are defined as above. For simulation, we choose $\alpha < 1/2$. The reason for choosing $\alpha \neq 1/2$ is that in the simulated setup, the strategy $[1/2]$ would asymptotically dominate the market and thus would push out any different strategy (β) . This is an immediate consequence of a Theorem by Amir et. al (2005), which states that – in this setting – the simple strategy $[1/2]$ will asymptotically overtake the market and thus prices become constant. Consequently for $\alpha = 1/2$, the return distribution becomes asymptotically a Dirac distribution with all its mass on 0. If $\alpha \neq 1/2$, the second strategy can coexist.

Figures 6-8 display the log return distributions for different parameters of the dynamical strategy: $\alpha = 0.4$ and $\beta = 0.2, 0.4, 0.473$. Each run has length 20,000, while the relative frequencies are estimated for a sample set of size 500. Simulations suggest that there exists a critical elasticity β^* such that the distribution for $|\beta^*|$ is Laplace, while for $\beta < |\beta^*|$, the graph of the distribution is a soft tent and approaches a parabola for small β .

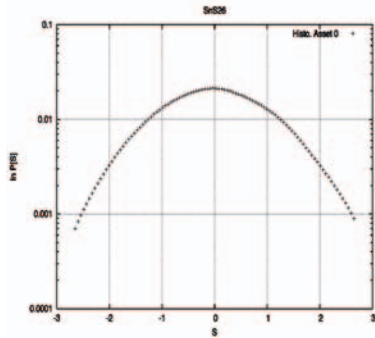


Figure 6: *Semilogarithmic plot of the return distribution for $[0.4], (+0.2)$*

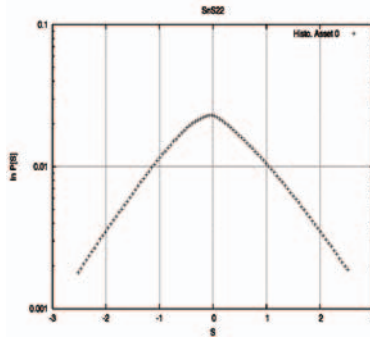


Figure 7: *Semilogarithmic plot of the theoretical return distribution for $[0.4], (+0.4)$*

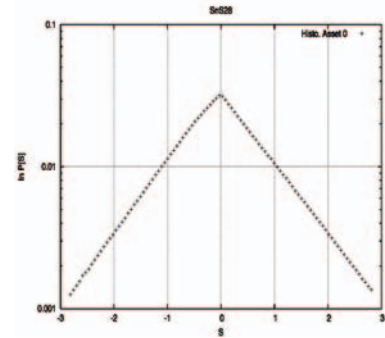


Figure 8: *Semilogarithmic plot of the return distribution for $[0.4], (+0.473)$*

Our simulations are compatible with the hypothesis by Madan and Seneta, who conjecture that asset returns are hyperbolically distributed. In particular, we see that that the shape of

the distribution depends on the parameters (α, β) . More precisely, given parameter α , the logarithmic return distribution looks like a parabola for small β , while if β is increased, then the graphs tends to a tent. The intuitive reason for this transition is the following: The larger the elasticity β is, the higher is the probability for large returns. Therefore, the larger the elasticity is, the more mass possess the tails in the distribution. This transition parallels the behavior of the hyperbolic distribution for increasing shape parameter ξ , where the distribution for $\xi = 1$ is a tent, while it is deformed into a parabola for $\xi \rightarrow 0$. Thus we say the return distribution from our model is qualitatively similar to an hyperbolic one.

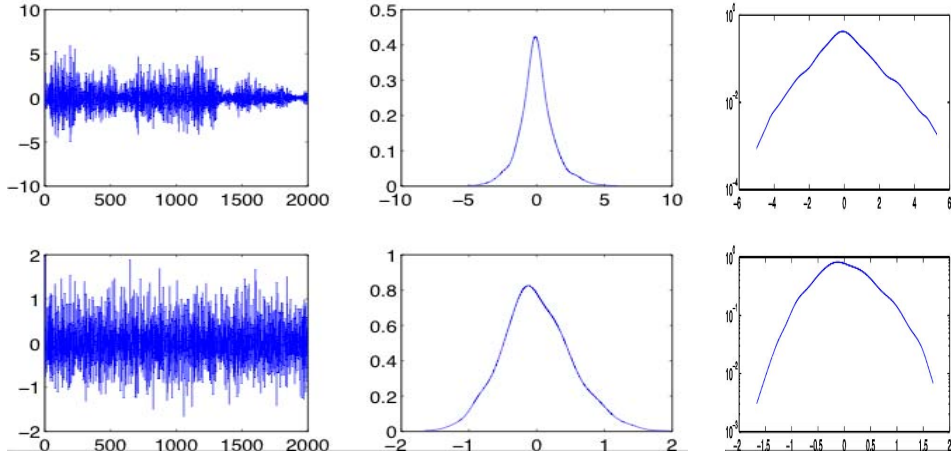


Figure 9: *Return trails for $\{ [0.4], (0.4) \}$ and the estimated kernel density (middle column) and also as a semi-logarithmic plot (right column). Row 1 is for the simulated return trail, while row two is the time series of the residuals obtained by correcting the original trails for volatility clustering.*

The graphs in Figure 9 are for parameter values $\alpha = 0.4$, $\beta = 0.4$. Our proposed model is obviously not a GARCH stochastic process, although the simulated returns exhibit the same type of volatility clustering as seen in real asset returns (and for which GARCH models can affectively model and predict). The unconditional return distribution, displayed in the upper left picture in Figure 9, shows the our elementary model exhibits pronounced volatility clustering, which is a standard feature of asset returns, and usually modelled with a GARCH process. As such, it makes sense to examine not only the unconditional distribution of the generated returns, but also their *conditional* distribution, for which we use a GARCH structure. In particular, we estimate a GARCH(1,1) model driven by iid hyperbolic disturbances. The parameters of the GARCH recursion and the two hyperbolic shape parameters are jointly estimated via maximum likelihood. For details on the stationarity conditions, computation, and performance

of such models, see Mittnik and Paoletta (2000). The conditional density of real asset data still exhibits heavy tails, but which are less fat than those of the unconditional density. In fact, the conditional distribution still is a tent-like graph, in the semi-logarithmic plot, and thus is hyperbolic-like itself. Moreover, the two figures to the very right show that the tails of the conditional distribution are less heavy than those of the unconditional distributions, as common in GARCH applications with real financial data.

The main concern of this note was to give economic evidence for the empirically reasonable conjecture that asset returns are hyperbolic-like distributed. We saw numerically that our economic model produces return distributions which are strongly non-Gaussian and exhibits semi-heavy tails and, thereby, even qualitatively resemble typical features common to the family of hyperbolic distributions. Moreover the returns distributions generated by our elementary model exhibit conditional heavy tails, i.e. after correcting for volatility clustering, the tails of the conditional distribution are hyperbolic-like, but less fat than those of the unconditional one. Returns have no significant autocorrelation for lags larger than 1, while simulated trails show volatility clustering. Thereby the autocorrelation of absolute returns decays to zero rapidly. Our model therefore reproduces important stylized facts derived from real asset data, while it supports the conjecture that asset returns are hyperbolic-like distributed.

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